Why Titles Matter: Influence and Optimal Hierarchy Depth

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Abstract

Organizations frequently rely on formal titles to structure influence in collective decision-making, especially when individual expertise is heterogeneous and difficult to observe directly. This paper studies how hierarchical titles can be designed to improve decision quality by shaping whose judgments carry greater authority. We show that titles act as an informational device: by summarizing imperfect evaluations of expertise, they allow organizations to concentrate decision authority on more reliable contributors, providing a rationale for the empirical correlation between higher rank and greater influence in decision-making. The analysis reveals a fundamental tradeoff in organizational design. While more intensive evaluation and finer rank differentiation improve decision accuracy, the informational gains from both dimensions exhibit sharply diminishing returns. As a result, when maintaining evaluations and hierarchy is costly, optimal organizations adopt finite screening and shallow hierarchies, with relatively simple title structures capturing most of the benefits of information aggregation.

Key words: Organizational design, Hierarchy depth, Information aggregation, Decision authority, Expertise, weighted majority

JEL codes: D23, D83, D71

1 Introduction

Modern organizations increasingly make important decisions in teams whose members have limited firsthand knowledge of one another's expertise. Employee–employer relationships are often short-lived—average job tenure in the U.S. private sector is under four years (U.S. Bureau of Labor Statistics, 2024)—and many firms rely on decentralized and team-based forms of collaboration that draw on specialized knowledge from different units (Bloom and Van Reenen, 2010; Bloom, Sadun, and Van Reenen, 2012b). At the same time, organizations frequently adjust their internal hierarchical structures and reporting relationships in response to technological change, market conditions, and competitive pressures, including changes in spans of control and the number of managerial layers (Rajan and Wulf, 2006; Bloom, Sadun, and Reenen, 2012a). Together, these features limit opportunities for long-term learning about coworkers' capabilities and make it common for individuals to collaborate with colleagues they have not worked with before.

Decision-making in such environments is particularly challenging. When collaborators lack reliable, experience-based information about one another's competence, they cannot easily assess whose judgments are more informative. Yet many organizational decisions are complex and high-stakes, so treating all opinions as equally reliable is rarely optimal. As a result, organizations face a fundamental design problem: how to structure influence within teams when expertise is heterogeneous but not directly observable.

In the absence of direct knowledge of expertise, organizations rely on institutional substitutes to guide collective decision-making. Formal titles—such as manager, director, or vice president—are especially prominent because they are publicly observable and organizationally endorsed. Although titles also govern reporting relationships and career incentives, they can be viewed as coarse summaries of prior evaluations of performance and ability. In practice, titles shape how information is aggregated: teams rarely treat all contributions symmetrically, and higher-ranked individuals tend to receive greater influence when opinions conflict or evidence is noisy. This creates an implicit aggregation rule in which rank determines decision authority, raising a natural design question: how finely should organizations differentiate titles, and how much information should they invest in doing so, in order to improve decision quality without incurring unnecessary organizational cost.

The study of hierarchy has deep roots in organizational economics. Early work emphasizes hierarchy as a response to bounded rationality and coordination constraints, allowing complex decision problems to be structured across organizational levels (Simon, 1947). Subsequent formal models characterize hierarchy as an information-processing architecture that economizes on communication and coordination costs in large organizations (Radner, 1993). More recent work focuses on three related roles of hierarchy: as a mechanism for allocating authority under communication frictions (Garicano and Prat, 2011), as an organizational response to incentive and monitoring problems (Garicano and Rossi-Hansberg, 2006), and as a determinant of managerial structure and spans of control in response to technological and competitive forces (Rajan and Wulf, 2006; Bloom et al., 2012a). Across these contributions, hierarchy serves a variety of organizational functions, but its role as a device for structuring information aggregation remains comparatively underexplored.

As a result, there is limited guidance on how the depth and granularity of hierarchy should be designed when titles primarily act as signals of expertise. Existing work provides little insight into how finely organizations should differentiate ranks, or how the informational gains from more detailed evaluations trade off against the organizational costs of maintaining a more complex hierarchy. When titles summarize imperfect assessments of ability, additional hierarchical layers can sharpen distinctions in expected expertise and improve decision quality, but they may also increase administrative burden and organizational complexity. Understanding this tradeoff requires treating hierarchy depth and evaluation intensity as explicit design choices in an information aggregation problem.

This paper studies a collective decision-making environment in which an organization aggregates assessments from a large number of individuals whose expertise is heterogeneous and not directly observable. Individuals observe noisy signals about a common underlying state, and the organization can invest in evaluations—such as tests, performance reviews, or certifications—that imperfectly reveal individual expertise. Evaluation outcomes are summarized by a finite set of observable titles, which serve as coarse indicators of expected expertise. Titles play no role in incentives, reporting lines, or communication; instead, they affect decisions only by shaping how individual assessments are weighted in the collective choice.

Building on standard results in the information aggregation literature (Nitzan and Paroush, 1982; Shapley and Grofman, 1984), we take the aggregation rule as given and focus on the organizational design problem. Using this framework, we study how decision accuracy responds to two design margins: the intensity of evaluation, which governs how informative titles are about expertise, and the depth of hierarchy, which determines how finely evaluation outcomes are translated into rank categories. We show that both margins improve decision quality by concentrating influence on more reliable individuals, but that their informational benefits exhibit sharply diminishing returns. Consequently, the optimal organizational design features a finite level of screening and a finite depth of hierarchy: relatively simple title structures capture most of the attainable gains from information aggregation, while further differentiation adds complexity with little additional benefit.

Related literature: This paper contributes to a large literature on organizational hierarchy by isolating and formalizing one specific function of hierarchical structure: its role in aggregating dispersed information when individual expertise is heterogeneous and imperfectly observed. A broad range of existing work studies hierarchy through other lenses, including bounded rationality, incentives and monitoring, authority allocation, communication, and problem solving. While these contributions provide deep insights into why hierarchies arise and how they shape organizational behavior, they typically abstract from the informational design problem that is central to our analysis.

As mentioned earlier, the study of hierarchy has deep roots in organizational economics.

Early work emphasizes hierarchy as an efficient response to bounded rationality and coordination constraints in complex organizations. Simon (1947) argues that hierarchical decomposition allows organizations to cope with cognitive limits by structuring complex decisions across multiple levels. Radner (1993) formalizes this perspective by modeling hierarchy as an information-processing architecture that economizes on communication and coordination costs in large organizations, with related contributions emphasizing the role of hierarchy in managing complexity and coordinating large systems (Marschak and Radner, 1972; Van Zandt, 1997). In this foundational literature, hierarchical structure is typically treated as an organizational primitive: the depth and granularity of hierarchy are taken as given, rather than as design variables chosen to improve decision accuracy.

A large body of work studies hierarchy as a mechanism for allocating authority, providing incentives, and mitigating monitoring problems within organizations. Classic models analyze how layered hierarchical structures address moral hazard, free riding, and effort provision in teams (Holmstrom, 1982; Mookherjee, 1984; McAfee and McMillan, 1991), while subsequent contributions examine how authority, ownership, and control interact with incentives under incomplete contracts (Baker, Gibbons, and Murphy, 1999, 2002). Related work emphasizes hierarchy as an allocation of decision rights under communication frictions, analyzing how authority and delegation shape organizational performance when information is costly to transmit or acquire (Bolton and Dewatripont, 1994; Aghion and Tirole, 1997). More recent theoretical and empirical contributions study how delegation, centralization, and managerial structure respond to communication constraints, technological change, and competitive pressures (Dessein, 2002; Rantakari, 2008; Alonso and Matouschek, 2008; Alonso, Dessein, and Matouschek, 2008; Bloom et al., 2012a). Across this literature, hierarchy primarily governs incentives, authority, and control. By contrast, we abstract entirely from incentives, monitoring, strategic communication, and delegation, and instead focus on hierarchy as an informational device that structures how dispersed assessments are aggregated into a collective decision.

A closely related strand of the literature studies how collective decisions aggregate heterogeneous information. Early contributions by Sah and Stiglitz (1986, 1988) compare hierarchical and polyarchical decision structures in terms of their error properties. Work by Nitzan and Paroush (1982) and Shapley and Grofman (1984) characterizes optimal weighting schemes for aggregating signals of unequal reliability, showing that differential influence can improve decision accuracy when information quality differs. Related results appear in models of voting and information aggregation with heterogeneous expertise (Condorcet, 1785; Austen-Smith and Banks, 1996). While these papers establish the value of unequal weighting, they typically take the aggregation structure as given and do not study how hierarchical categories or titles should be designed endogenously, nor how the depth of hierarchy trades off against organizational costs.

A related strand of the literature studies hierarchy as a mechanism for allocating problem-solving tasks and specialized knowledge. In influential work, Garicano (2000) models organizations as knowledge hierarchies in which lower-level agents handle routine problems while rare or complex cases are escalated to more knowledgeable specialists. Subsequent research examines how communication technology and cognitive constraints shape optimal hierarchical depth and expertise specialization (Garicano and Rossi-Hansberg, 2006; Garicano and Prat, 2011), as well as related models of problem routing and knowledge allocation (Beggs, 2001). In this literature, hierarchy governs the assignment and escalation of problems across levels, whereas we study how hierarchy shapes the aggregation of many simultaneous assessments into a single collective decision.

Our paper complements these literatures by studying hierarchy exclusively as a mechanism for structuring information aggregation when individual expertise is heterogeneous and imperfectly observed. We model formal titles as coarse summaries of noisy evaluations of expertise and allow organizations to choose both how much information to generate about expertise and how finely to translate that information into hierarchical categories. Unlike existing work, hierarchy in our model does not govern reporting lines, delegation, incentives, or monitoring, and communication is costless conditional on titles. By endogenizing the depth of hierarchy and the assignment of evaluations to titles, we characterize when additional layers of hierarchy improve decision accuracy and when their marginal informational value is outweighed by organizational costs. This perspective highlights an additional and distinct function of hierarchy—information filtering—that organizations must consider alongside incentives, authority, and coordination when designing hierarchical structures.

The remainder of the paper is organized as follows. Section 2 presents the model: a binary decision problem with heterogeneous and unobservable expertise, a noisy evaluation technology, and the induced title structure. Section 3 analyzes decision-making under a fixed evaluation-and-title system, characterizes the optimal title-based aggregation rule, and derives a closed-form expression for the asymptotic error exponent that summarizes decision performance. Section 4 studies the organization's design problem, examining how performance changes with evaluation intensity and hierarchy depth and characterizing optimal choices under cost constraints. Section 5 discusses the implications of the results for organizational design and hierarchy. Section 6 concludes.

2 Model Setup

We develop a model of organizational decision-making under uncertainty in which an organization aggregates information from multiple agents whose expertise is heterogeneous and not directly observable. Agents provide assessments that differ in reliability, and the organization uses observable titles—generated through imperfect evaluation—to structure decision authority when forming a collective choice. The model specifies the state space, agents' information, the evaluation and title-assignment process, and the aggregation rule used to reach a decision.

State and Decision

The organization faces a binary decision regarding a project or action. The underlying state of the world is denoted by $\omega \in \{-1, 1\}$, where the labels -1 and 1 are purely notational. The organization holds a symmetric prior,

$$\Pr(\omega = 1) = \Pr(\omega = -1) = \frac{1}{2},$$

so any improvement in decision quality must come from internal assessments.

After collecting assessments from its members, the organization chooses an action

$$d \in \{-1, 1\}.$$

The organization aims to match the realized state, and it does so through an internal decision protocol that specifies how individual assessments are aggregated; this protocol is fixed ex ante.

The organization's payoff depends on whether the chosen action matches the state. Specifically,

$$\pi(d, \omega) = \begin{cases} 1, & \text{if } d = \omega, \\ 1 - \kappa, & \text{if } d \neq \omega, \end{cases}$$

where $\kappa > 0$ measures the loss from an incorrect decision.

To capture that decision stakes may rise with project scale and may vary across projects, we let

$$\kappa = \lambda^n,$$

where n indexes the scale or complexity of the project and $\lambda > 1$ is a random variable capturing the severity of losses conditional on failure. We assume that λ is drawn from a uniform distribution on $[1, 2]$, independently of all other primitives. The realization of

λ reflects project-specific factors (e.g., technological risk, regulatory exposure, or market conditions) and is observed by the organization before the final action is taken.

Finally, the organization may choose not to proceed with a project when its expected payoff is negative. Throughout, expected performance is evaluated relative to an outside option normalized to zero.

Workers and Information

The organization relies on assessments produced by its members. There are n workers indexed by $i = 1, 2, \ldots, n$, each of whom generates a binary assessment relevant to the decision. Workers differ in their expertise, which affects the reliability of the information they produce.

Worker i has an ability level $\theta_i \in (\frac{1}{2}, 1)$. Ability is not directly observable to the organization or to other workers. Conditional on the underlying state ω, worker i produces an assessment $s_i \in \{-1, 1\}$, where

$$\Pr(s_i = \omega \mid \omega, \theta_i) = \theta_i.$$

Each assessment is therefore informative about the underlying state, though its reliability varies across workers.

Conditional on the state, the assessments $\{s_i\}_{i=1}^n$ are independent. Each assessment represents primary information generated through task execution, analysis, or judgment, rather than being filtered through organizational hierarchy. We abstract from incentive provision and strategic communication in order to focus on how organizations aggregate heterogeneous information when individual expertise is not directly observable.

The ability levels $\{\theta_i\}_{i=1}^n$ are drawn independently from a common distribution F with density f, supported on $(\frac{1}{2}, 1)$. The distribution F is known to the organization, but the individual ability levels $\{\theta_i\}_{i=1}^n$ are not observed.

Evaluation and Titles

Before aggregating assessments, the organization may invest in evaluating workers' expertise. The evaluation technology is a test consisting of m independent items. Conditional on ability θ_i, worker i answers each item correctly with probability θ_i, independently across items. Let $J_i \in \{0, 1, \ldots, m\}$ denote worker i's number of correct answers. Then

$$J_i \sim \text{Binomial}(m, \theta_i).$$

The organization observes J_i but not θ_i.

Based on the test score, the organization assigns each worker to one of L title categories. An assignment rule is a deterministic map

$$r : \{0, 1, \ldots, m\} \to \{1, 2, \ldots, L\},$$

and worker i's title is

$$K_i := r(J_i) \in \{1, 2, \ldots, L\}.$$

Titles are publicly observable and are interpreted as coarse indicators of expected expertise.

The assignment rule induces, for each category $k \in \{1, \ldots, L\}$, the population share p_k, the posterior density f_k, and the corresponding mean ability ξ_k, defined by

$$p_k := \Pr(K_i = k), \qquad f_k(\theta) := f(\theta \mid K_i = k), \qquad \xi_k := \mathbb{E}[\theta_i \mid K_i = k].$$

These category-level objects summarize how the evaluation and title system compresses information about expertise.

Implementing the evaluation and title system entails organizational costs, summarized by a cost function

$$C = C(m, L),$$

which depends on the intensity of evaluation m and the number of title categories L. We assume that the cost is weakly increasing in both arguments, so that $C_m(\cdot) \geq 0$ and $C_L(\cdot) \geq 0$. The cost reflects resources devoted to administering evaluations, processing information, and maintaining a more differentiated title structure. No further structure is imposed in the model setup.

Aggregation Rule

The organization aggregates the information generated by its members to form a collective decision. Individual assessments provide noisy information about the underlying state, and observable titles affect how these assessments are incorporated into the final choice.

Formally, the organization's decision rule maps the profile of assessments into an action. An aggregation rule is therefore written as

$$d = \delta(s_1, \ldots, s_n;\, K_1, \ldots, K_n),$$

where δ is a function from assessment profiles to a binary action, conditional on the realized titles.

The aggregation rule captures how the organization combines information when forming a collective decision. Titles enter as parameters that shape the relative influence of individual assessments, while the assessments themselves are the primary carriers of information. The characterization of optimal aggregation rules and the induced decision authority across titles is developed in the analysis.

Organizational Design Problem

The organization chooses its internal decision structure to maximize expected performance net of organizational costs. A design specifies how expertise is evaluated and incorporated into collective decisions.

Given a design (m, L, r, δ) and a realization of λ, the organization's expected payoff from undertaking the project is $\mathbb{E}[\pi(d, \omega) \mid \lambda]$, where the expectation is taken over the state, workers' abilities, assessments, and induced titles. The organization may choose to discontinue the project if its expected value is negative. As a result, the ex ante value of a design is given by $\mathbb{E}_\lambda[\max\{\mathbb{E}[\pi(d, \omega) \mid \lambda], 0\}]$.

The organizational design problem is

$$\max_{m,L,r,\delta} \mathbb{E}_\lambda\Big[\max\big\{\mathbb{E}[\pi(d,\omega) \mid \lambda],\, 0\big\}\Big] \;-\; C(m, L).$$

The aggregation rule δ determines how individual assessments are combined given the realized titles, while the evaluation and title system—summarized by the tuple (m, L, r)—governs how much information about expertise is revealed and how it is coarse-grained into observable ranks.

3 Analysis

This section analyzes the organization's decision-making problem given an evaluation-and-title system. We first characterize the aggregation rule that maximizes the probability of a correct collective decision. We then derive a tractable measure of decision performance and use it to study the organization's choice of (m, L, r). Finally, we present benchmark cases that bound achievable performance and discipline the design problem.

3.1 Optimal Aggregation Rule

Fix an evaluation-and-title system (m, L, r). The evaluation and title assignment induce category shares $\{p_k\}_{k=1}^L$ and posterior mean abilities $\{\xi_k\}_{k=1}^L$, where $\xi_k = \mathbb{E}[\theta_i \mid K_i = k]$. Conditional on the state, individual assessments are independent and satisfy

$$\Pr(s_i = \omega \mid \omega, K_i = k) = \xi_k.$$

Thus, conditional on titles, workers within the same category are observationally symmetric from the organization's perspective. The aggregation problem therefore reduces to choosing a category-dependent influence weight for each assessment.

Among all decision rules that map $(s_1, \ldots, s_n; K_1, \ldots, K_n)$ into an action $d \in \{-1, 1\}$, an optimal rule takes a likelihood-ratio form.

Lemma 1 (Optimal title-based aggregation)**.** *To maximize the probability of a correct collective decision,* $\Pr(d = \omega)$*, an optimal aggregation rule is a weighted majority rule:*

$$d = \operatorname{sgn}\left(\sum_{i=1}^{n} w_{K_i}\, s_i\right),$$

where the optimal weight for category k is the log-odds ratio

$$w_k = \log\left(\frac{\xi_k}{1 - \xi_k}\right).$$

When the weighted sum equals zero, the organization randomizes uniformly over $\{-1, 1\}$.

Lemma 1 is a standard likelihood-ratio aggregation result in weighted voting with heterogeneous reliabilities (see, e.g., Nitzan and Paroush, 1982; Shapley and Grofman, 1984). Intuitively, the rule places more weight on titles associated with higher posterior reliability, and the log-odds form is the natural weight under conditional independence.

Lemma 1 reduces the aggregation problem to the category objects induced by the evaluation and title system; in particular, the optimal weights depend on the title system only through the posterior means $\{\xi_k\}_{k=1}^{L}$. Equivalently, w_k is the log-likelihood-ratio contribution of a category-k assessment under conditional independence.

3.2 Decision Performance and Organizational Design

Applying the optimal aggregation rule in Lemma 1 yields a tractable representation of decision performance. The key object is an error exponent that summarizes how quickly the probability of a wrong decision vanishes as the number of workers grows.

Fix (m, L, r) and consider the weighted majority rule in Lemma 1. Let

$$\Pr_n(m, L, r) := \Pr(d \neq \omega \mid m, L, r)$$

denote the error probability with n workers under this rule, where the probability is taken over $(\omega, \{\theta_i\}_{i=1}^n, \{J_i\}_{i=1}^n, \{K_i\}_{i=1}^n, \{s_i\}_{i=1}^n)$.

Definition 1 (Error exponent)**.** *The (asymptotic) error exponent is*

$$\phi(m, L, r) \ := \ -\lim_{n\to\infty} \frac{1}{n} \log \Pr_n(m, L, r),$$

whenever the limit exists.

When the error exponent exists, the probability of a wrong decision satisfies

$$\Pr_n(m, L, r) = e^{-n\phi(m,L,r)} \quad \text{as } n \to \infty.$$

A larger $\phi(m, L, r)$ corresponds to faster exponential decay of the error probability as team size increases. Under the optimal title-based weights, $\phi(m, L, r)$ admits a closed-form expression in terms of the category objects $(p_k, \xi_k)_{k=1}^L$ induced by the evaluation and title system.

Proposition 1 (Error exponent under optimal title-based weights)**.** *Under the weighted majority rule in Lemma 1, the error exponent satisfies*

$$\phi(m, L, r) = -\log\left[\sum_{k=1}^{L} 2p_k \sqrt{\xi_k\big(1-\xi_k\big)}\right].$$

Proposition 1 implies that, for a fixed evaluation-and-title system, decision performance depends on (m, L, r) only through the induced category objects $(p_k, \xi_k)_{k=1}^L$.

We now express the organization's design objective in terms of $\phi(m, L, r)$. Recall that the loss from an incorrect decision is $\kappa = \lambda^n$, where λ is drawn uniformly from $[1, 2]$ and observed by the organization before choosing the action. Under the weighted rule, the organization's expected payoff conditional on λ is

$$\mathbb{E}[\pi(d, \omega) \mid \lambda] = 1 - \lambda^n \Pr_n(m, L, r).$$

Using Proposition 1, this becomes

$$\mathbb{E}[\pi(d, \omega) \mid \lambda] = 1 - \lambda^n e^{-n\phi(m,L,r)}.$$

As $n \to \infty$, the sign of $\log \lambda - \phi(m, L, r)$ determines whether the project has positive value. With the outside option normalized to zero, the organization proceeds if and only if $\lambda \leq e^{\phi(m,L,r)}$. Therefore, the ex ante value of design (m, L, r) is

$$\mathbb{E}_\lambda[\max\{\mathbb{E}[\pi(d, \omega) \mid \lambda], 0\}] = e^{\phi(m,L,r)} - 1,$$

for $\phi(m, L, r) \in (0, \log 2)$, since $\lambda \sim U[1, 2]$ and for $e^\phi \in (1, 2)$, the probability is $e^\phi - 1$.

The organizational design problem therefore reduces to choosing (m, L, r) to maximize performance net of organizational costs:

$$\max_{m,L,r} \; e^{\phi(m,L,r)} - 1 - C(m, L).$$

3.3 Benchmarks

Benchmarks discipline the organizational design problem by providing bounds on achievable performance and clarifying the role of evaluation and hierarchy in the model.

First-best benchmark. Suppose individual expertise θ_i is publicly observed and can be used directly in aggregation, so that no evaluation or title system is required and no organizational cost is incurred. In this case, the organization assigns each assessment the weight

$$w_i = \log\left(\frac{\theta_i}{1-\theta_i}\right),$$

and aggregates assessments using the corresponding likelihood-ratio rule. Since abilities are observed, the aggregation rule fully exploits all available information.

Let ϕ^{FB} denote the resulting error exponent. By the law of large numbers and standard large-deviation arguments, the error exponent satisfies

$$\phi^{FB} = -\log\left[\int_{1/2}^{1} 4\sqrt{\theta(1-\theta)}\, dF(\theta)\right],$$

where F is the cross-sectional distribution of expertise. The associated value is

$$U^{FB} = e^{\phi^{FB}} - 1 = \left[\int_{1/2}^{1} 4\sqrt{\theta(1-\theta)}\, d\theta\right]^{-1} - 1 = \frac{4}{\pi} - 1 \approx 0.273.$$

This benchmark represents the maximal performance attainable in the model, since any feasible design must rely on imperfect information about individual expertise.

No-evaluation benchmark. At the opposite extreme, consider a design with no evaluation and a single title category ($m = 0,\ L = 1$). In this case, all workers are treated symmetrically and receive the same aggregation weight. The induced posterior mean ability is $\xi = \mathbb{E}[\theta_i]$, and the error exponent reduces to

$$\phi^{NE} = -\log\left[2\sqrt{\xi(1-\xi)}\right].$$

When $\theta_i \sim U[1/2, 1]$, the mean ability is $\xi = 3/4$, so the corresponding value is

$$U^{NE} = \left[2\sqrt{\xi(1-\xi)}\right]^{-1} - 1 = \left[2\sqrt{\tfrac{3}{16}}\right]^{-1} - 1 = \left[2 \cdot \frac{\sqrt{3}}{4}\right]^{-1} - 1 = \frac{2}{\sqrt{3}} - 1 \approx 0.155.$$

This benchmark captures decision-making without internal differentiation of expertise and serves as a lower bound on performance when evaluation and hierarchy are absent.

Together, these benchmarks delimit the scope for organizational design. Evaluation and hierarchy improve performance by moving the organization away from the no-evaluation benchmark toward the first-best bound, but cannot surpass the latter. The design problem studied in this paper concerns how much of this gap can be closed, and at what organizational cost.

4 Results

This section presents the main results of the analysis. We examine how decision performance varies with the organization's design choices and report the implications of changes in evaluation intensity and hierarchical structure. The section combines analytical characterizations with numerical illustrations to describe how performance responds to different designs (m, L, r).

4.1 Performance Without Organizational Costs

We begin by examining organizational performance in the absence of organizational costs, so that $C(m, L) = 0$. This benchmark isolates the informational value of evaluation and hierarchy and clarifies the shape of the performance function before introducing tradeoffs.

Figure 1 plots performance as a function of hierarchy depth L holding evaluation intensity fixed at $m = 300$. Increasing the number of title categories substantially improves performance at low levels of hierarchy, but the gains flatten quickly. Beyond a modest number of

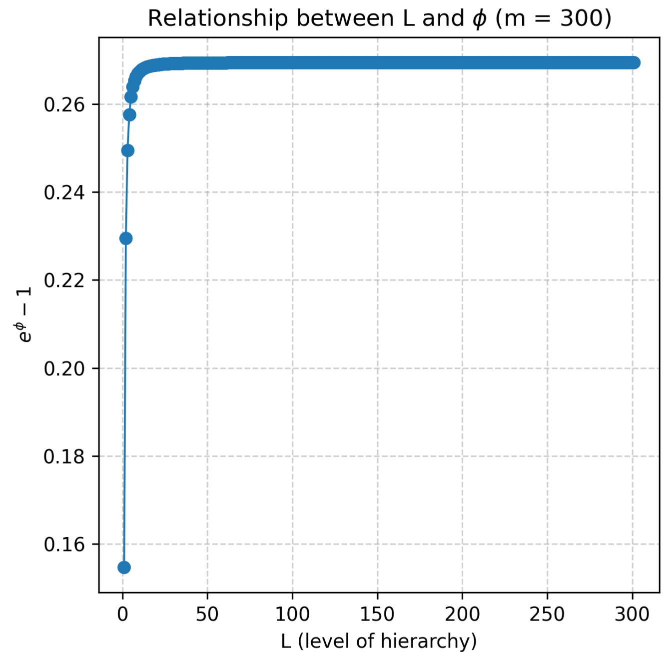


Figure 1: Organizational performance as a function of hierarchy depth L for fixed evaluation intensity $m = 300$.

levels, additional differentiation yields only marginal improvements. Performance converges to an upper bound corresponding to the first-best benchmark derived in Section 3.3.

Figure 2 shows performance as evaluation intensity increases when the number of title categories grows with the number of tests according to $L = m + 1$. Performance rises rapidly at low levels of testing and then exhibits pronounced concavity, approaching the same upper bound. This case represents an extreme form of organizational differentiation in which increasingly fine evaluations are translated into correspondingly fine title distinctions.

To disentangle the roles of evaluation and hierarchy, Figure 3 plots performance as a function of evaluation intensity m for fixed hierarchy depths $L \in \{5, 10, 50\}$. Across all cases, performance is increasing and concave in m, with diminishing marginal gains from additional testing. Holding L fixed limits the maximal attainable performance, but even coarse hierarchies capture a substantial fraction of the available gains once evaluation is sufficiently informative.

Taken together, these results highlight two robust patterns. First, both evaluation in-

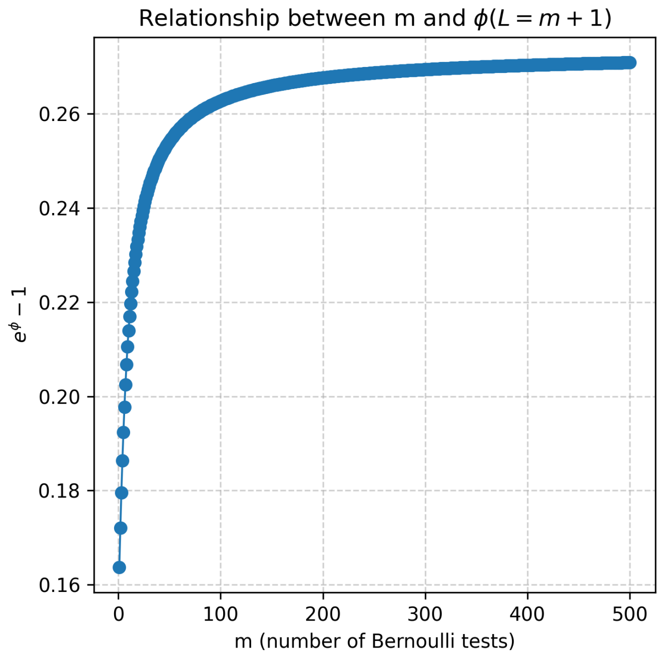
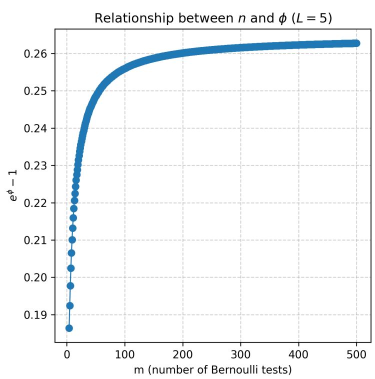
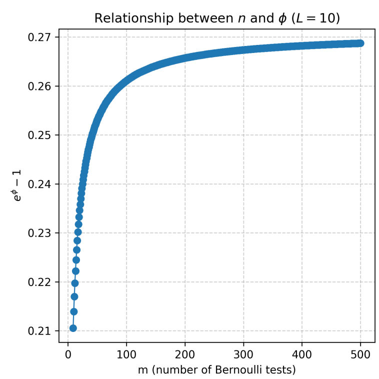


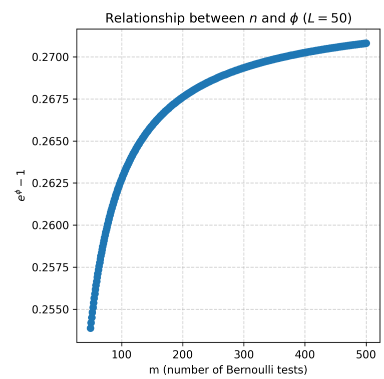
Figure 2: Organizational performance as evaluation intensity m increases when hierarchy depth grows according to $L = m + 1$.



(a) $L = 5$



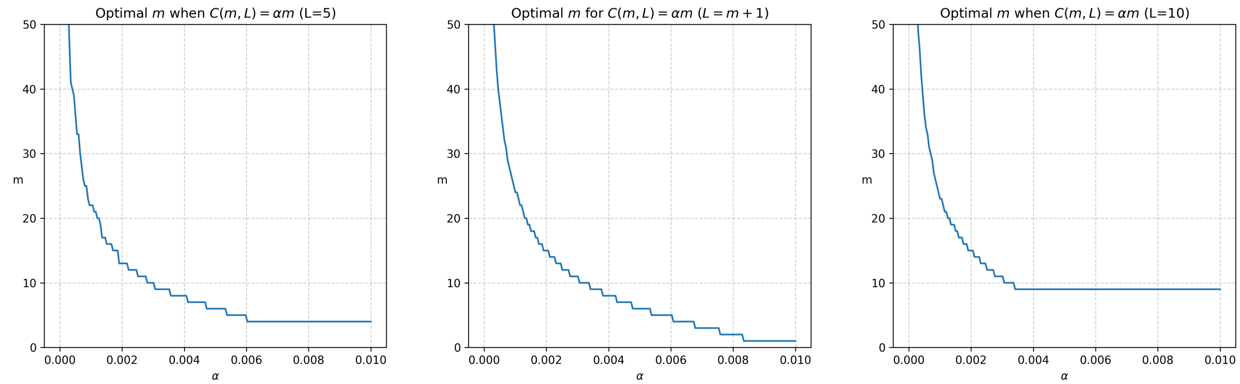
(b) $L = 10$



(c) $L = 50$

Figure 3: Organizational performance as a function of evaluation intensity m for fixed hierarchy depths $L \in \{5, 10, 50\}$.

tensity and hierarchy depth improve decision performance by sharpening the allocation of decision authority toward more reliable contributors. Second, the informational returns to both dimensions are strongly diminishing, and performance converges rapidly toward its upper bound. These features foreshadow the emergence of finite optimal designs once organizational costs are introduced.



(a) $L = 5$ (b) $L = m + 1$ (c) $L = 10$

Figure 4: Optimal evaluation intensity m^* as a function of the marginal testing cost α under linear evaluation costs.

4.2 Optimal Evaluation Intensity

We now introduce organizational costs and study how they shape the optimal amount of evaluation. Throughout this subsection, the cost of evaluation is linear in the number of tests,

$$C(m, L) = \alpha m,$$

where $\alpha > 0$ measures the marginal cost of administering an additional test. For each value of α, the organization chooses m to maximize

$$U(m, L) = e^{\phi(m,L)} - 1 - \alpha m,$$

taking the hierarchy depth L as given. For each (m, L), the assignment rule r is optimized.

Figure 4 plots the optimal number of tests m^* as a function of α under three organizational environments. The left panel fixes hierarchy depth at $L = 5$, the middle panel considers the case in which hierarchy expands with evaluation according to $L = m+1$, and the right panel fixes hierarchy depth at $L = 10$.

Across all cases, the optimal evaluation intensity is decreasing in the marginal cost α. When testing is inexpensive, the organization invests heavily in evaluation to improve the allocation of decision authority across titles. As α increases, the marginal informational benefit of additional testing is outweighed by its direct cost, leading to discrete reductions in m^*. These stepwise patterns reflect both the integer nature of the design problem and the diminishing returns to evaluation documented in Section 4.1.

4.3 Optimal Hierarchy Depth

We next examine how the optimal depth of hierarchy responds to organizational costs. In this subsection, the cost function takes the form

$$C(m, L) = \beta L,$$

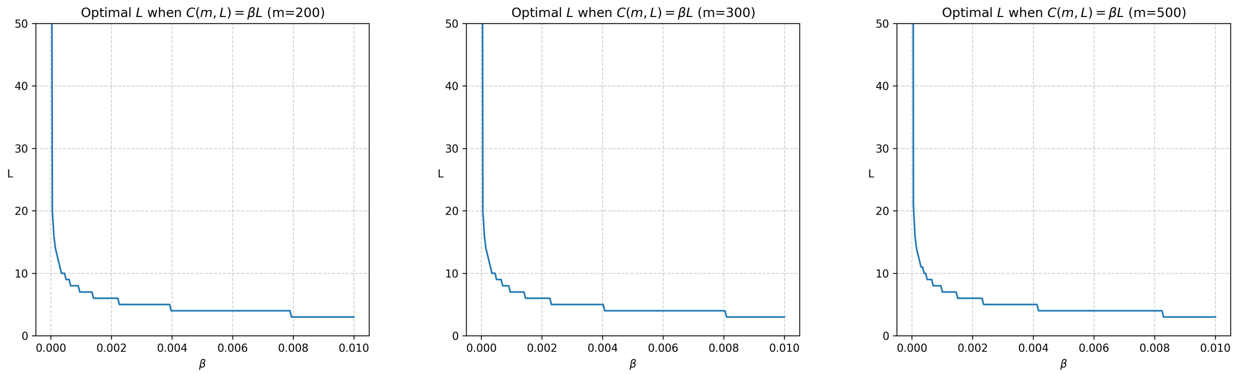
where $\beta > 0$ captures the marginal cost of maintaining an additional hierarchical level. Evaluation intensity m is fixed, and for each (β, m) the organization chooses the hierarchy depth L to maximize

$$U(m, L) = e^{\phi(m,L)} - 1 - \beta L,$$

with the assignment rule r optimized for each (m, L).

Figure 5 plots the optimal hierarchy depth L^* as a function of β for different fixed values of evaluation intensity m. Across all cases, the optimal hierarchy depth declines sharply as the marginal cost of hierarchy increases. When β is small, the organization adopts a relatively deep hierarchy in order to exploit fine distinctions in expected expertise. As β rises, the informational gains from additional levels are quickly dominated by organizational costs, leading to discrete reductions in L^*.

Two patterns are worth emphasizing. First, the optimal hierarchy depth is highly sensitive to even modest increases in β: most of the reduction in L^* occurs at low values of the marginal cost. Second, higher evaluation intensity supports deeper hierarchies, but only when the cost of hierarchy is sufficiently low. Once β exceeds a moderate threshold, the optimal structure collapses to a shallow hierarchy regardless of how informative evaluation



(a) $m = 200$ (b) $m = 300$ (c) $m = 500$

Figure 5: Optimal hierarchy depth L^* as a function of the marginal hierarchy cost β, for selected values of evaluation intensity m.

is.

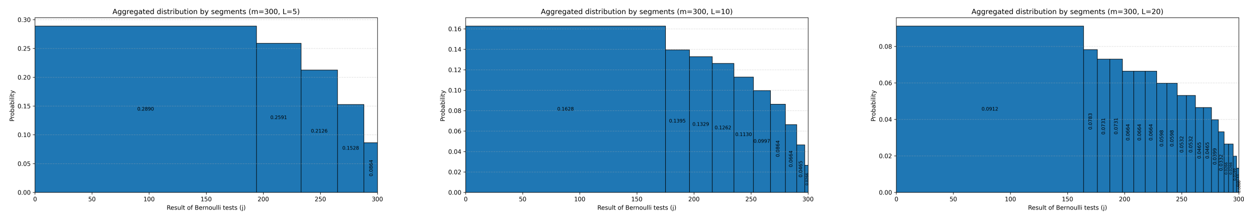
Together, these results highlight that hierarchy depth is the most fragile dimension of organizational design. While evaluation intensity can be adjusted smoothly in response to costs, the optimal number of hierarchical levels responds discretely and declines rapidly as coordination and communication costs increase.

4.4 Structure of the Optimal Assignment Rule

We focus on coarser hierarchies that are empirically more relevant and visually informative. The figures below display the aggregated assignment rules for $L = 5$, $L = 10$, and $L = 20$, holding evaluation intensity fixed at $m = 300$.

Figure 6 shows that, for $L = 5$, $L = 10$, and $L = 20$, the optimal assignment rule groups wide ranges of test outcomes into a small number of title categories. Lower titles pool a large mass of relatively uninformative evaluations, while higher titles are reserved for increasingly extreme test outcomes. As L increases, the partition becomes progressively finer, but the additional cuts occur primarily in the upper tail of the evaluation distribution.

Two patterns emerge. First, optimal assignment rules are monotone in evaluation outcomes: higher test scores are always mapped to weakly higher titles. Second, hierarchy depth primarily affects how finely the upper tail of the evaluation distribution is separated. Addi-



(a) $L = 5$ (b) $L = 10$ (c) $L = 20$

Figure 6: Aggregated assignment rules for $m = 300$ under different hierarchy depths.

tional title categories are used to refine distinctions among high-performing workers, while lower-performing workers remain pooled into broad categories. This reflects diminishing returns to differentiation at low levels of expected expertise, where additional information has little effect on decision quality.

Overall, these results confirm that the role of the assignment rule is largely supportive. Once the aggregation rule is optimally chosen, the main determinants of performance are evaluation intensity and hierarchy depth. The precise shape of the assignment rule matters primarily insofar as it governs how finely high levels of expertise are distinguished, which explains why changes in r have smaller quantitative effects than changes in m or L.

5 Discussion

This paper studies hierarchy as a mechanism for allocating decision authority when individual expertise is heterogeneous and imperfectly observed. Titles summarize noisy evaluations of expertise and determine how individual assessments are weighted in collective decisions. The analysis highlights how organizational performance depends on three design choices: evaluation intensity, hierarchy depth, and the assignment of evaluations to titles.

A central finding is that both evaluation and hierarchy improve decision quality, but with strongly diminishing returns. More intensive evaluation sharpens posterior beliefs about expertise, while additional title categories allow the organization to translate these beliefs into finer distinctions in decision authority. However, once the most informative distinctions are already captured, further refinement yields little additional benefit. As a result, performance converges quickly toward an upper bound well below the first-best benchmark. This explains why relatively simple organizational structures can perform nearly as well as much more complex ones.

The results also reveal an important asymmetry between evaluation intensity and hierarchy depth. Evaluation can be adjusted relatively smoothly in response to costs, whereas hierarchy depth responds discretely. Small increases in the marginal cost of hierarchy lead to sharp reductions in the optimal number of title categories. This reflects the fact that hierarchy primarily refines distinctions among high-expertise workers: when hierarchy is costly, the organization optimally collapses many intermediate ranks while preserving a small number of top positions. In this sense, hierarchy depth is the most fragile dimension of organizational design.

The analysis further clarifies the role of the assignment rule. While the optimal rule is monotone and respects evaluation rankings, its detailed shape plays a secondary role once aggregation weights are optimally chosen. Changes in performance are driven mainly by how many distinctions the organization makes, not by the exact placement of thresholds. This helps explain why the numerical structure of assignment rules varies across environments without substantially affecting performance. From a design perspective, this suggests that organizations can rely on relatively simple and robust assignment procedures without sacrificing much efficiency.

Several limitations of the analysis point to directions for future work. First, the model is static and abstracts from learning and promotion over time. In practice, titles evolve as workers accumulate performance histories, and hierarchical structures may serve additional incentive or career concerns. Second, the evaluation technology is stylized as a sequence of independent Bernoulli tests. Other screening mechanisms—such as peer evaluation, project outcomes, or task-specific signals—may interact differently with hierarchy. Finally, the model treats communication costs in reduced form through the cost of hierarchy, rather than modeling explicit information loss across layers.

Despite these simplifications, the framework captures a fundamental tradeoff in organizational design. Hierarchy improves decision-making by concentrating authority among more reliable contributors, but only up to the point where organizational costs outweigh informational gains. The results provide a rationale for why many organizations adopt shallow hierarchies with limited differentiation, and why attempts to excessively refine internal rank structures often yield little improvement in decision quality.

6 Conclusion

This paper studies how organizations should design internal hierarchies when individual expertise is heterogeneous, imperfectly observed, and must be aggregated into a collective decision. Titles arise as a coarse summary of noisy evaluations of expertise and serve to allocate decision authority among contributors. By embedding hierarchy into a formal information-aggregation framework, the analysis clarifies how evaluation, rank differentiation, and collective decision-making interact.

The model delivers three main insights. First, both evaluation intensity and hierarchy depth improve decision accuracy by concentrating influence on more reliable contributors, but the gains exhibit strong diminishing returns. As a result, relatively simple organizational structures capture most of the achievable improvements in decision quality. Second, hierarchy depth responds sharply to organizational costs. Even modest increases in the marginal cost of hierarchy lead to substantial reductions in the optimal number of title categories, whereas evaluation intensity adjusts more smoothly. This makes hierarchy depth the most fragile dimension of organizational design. Third, while the optimal assignment of evaluations to titles is monotone and intuitive, its detailed shape plays a secondary role once aggregation weights are optimally chosen. Performance is driven primarily by how many distinctions the organization makes, rather than by the precise placement of thresholds.

Together, these results provide a disciplined explanation for why many organizations rely on shallow hierarchies with limited rank differentiation, even when decision-making is complex and expertise is unevenly distributed. Hierarchy improves decision quality by filtering noise and concentrating authority, but only up to the point where organizational costs outweigh informational gains. Beyond that point, additional layers contribute little and may be counterproductive.

More broadly, the framework highlights the informational role of hierarchy distinct from its incentive or control functions. By viewing titles as a mechanism for compressing information about expertise and structuring influence, the analysis offers a new perspective on organizational design in environments where direct knowledge of ability is limited and collaboration is transient.

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A Proofs

Proof of Proposition 1. From the main text, we know that $\phi(m, L, r)$ under a fixed evaluation-and-title system is

$$\phi(m, L, r) = -\lim_{n\to\infty} \frac{1}{n} \log \Pr_n(m, L, r) \quad \text{where } \Pr_n(m, L, r) = \Pr(d \neq \omega \mid m, L, r).$$

The key is to pin down $\Pr(d \neq \omega \mid m, L, r)$ under the weighted majority rule stated in Lemma 1. Recall that for each category $k \in \{1, \dots, L\}$, the population share p_k, the posterior density f_k, and the corresponding mean ability ξ_k, defined by

$$p_k = \Pr(K_i = k), \qquad f_k(\theta) = f(\theta \mid K_i = k), \qquad \xi_k = \mathbb{E}[\theta_i \mid K_i = k].$$

Without loss of generality, we assume that $\omega = 1$ and $\xi_1 < \xi_2 < ... < \xi_L$. Now denote $w_k = \log \frac{\xi_k}{1-\xi_k}$, and X_i as the random variable of the weighted vote of agent i, where

$$X_i = \begin{cases} w_{K_i} & s_i = 1 \\ -w_{K_i} & s_i = -1 \end{cases}.$$

We can further write X_i as

$$X_i = \begin{cases} w_1 & \text{with probability } p_1\xi_1 \\ -w_1 & \text{with probability } p_1(1-\xi_1) \\ w_2 & \text{with probability } p_2\xi_2 \\ -w_2 & \text{with probability } p_2(1-\xi_2) \\ \dots & \dots \\ w_L & \text{with probability } p_L\xi_L \\ -w_L & \text{with probability } p_L(1-\xi_L) \end{cases}.$$

Let $S_n = X_1 + X_2 + ... + X_n$, then the decision is $d = 1$ if $S_n > 0$, $d = -1$ if $S_n < 0$, and random when $S_n = 0$. Now by the weighted majority rule stated in Lemma 1, we know that

$$\Pr(d \neq \omega) = \Pr(S_n < 0) + \frac{1}{2}\Pr(S_n = 0).$$

As we will show in the later proof, $\lim_{n\to\infty} \frac{1}{n}\log\Pr(S_n < 0) = \lim_{n\to\infty} \frac{1}{n}\log\Pr(S_n \leq 0)$. Thus, we can conclude that

$$\phi(m, L, r) = -\lim_{n\to\infty} \frac{1}{n}\log\Pr(S_n < 0).$$

Moment generating function

Calculate the moment generating function:

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{k=1}^{L} p_k \left(\xi_k e^{w_k t} + (1-\xi_k)e^{-w_k t}\right).$$

Lemma 2. *$M(t)$ is strictly convex, and so is $\log M(t)$.*

Proof. Taking derivatives, we obtain

$$M'(t) = \sum_{k=1}^{L} p_k w_k \Big(\xi_k e^{w_k t} - (1-\xi_k)e^{-w_k t}\Big),$$

$$M''(t) = \sum_{k=1}^{L} p_k w_k^2 \Big(\xi_k e^{w_k t} + (1-\xi_k)e^{-w_k t}\Big) > 0.$$

Thus $M(t)$ is strictly convex. Moreover,

$$\frac{d\log M(t)}{dt} = \frac{M'(t)}{M(t)}, \qquad \frac{d^2\log M(t)}{dt^2} = \frac{M''(t)M(t) - (M'(t))^2}{(M(t))^2}.$$

Let

$$a_k = \sqrt{p_k\Big(\xi_k e^{w_k t} + (1-\xi_k)e^{-w_k t}\Big)}, \qquad b_k = w_k a_k.$$

By the Cauchy–Schwarz inequality,

$$M''(t)M(t) = \Big(\sum_{k=1}^{L} a_k^2\Big)\Big(\sum_{k=1}^{L} b_k^2\Big) \geq \Big(\sum_{k=1}^{L} a_k b_k\Big)^2.$$

We also have

$$\Big(\sum_{k=1}^{L} a_k b_k\Big)^2 = \Big(\sum_{k=1}^{L} p_k w_k \big(\xi_k e^{w_k t} + (1-\xi_k) e^{-w_k t}\big)\Big)^2 > \Big(\sum_{k=1}^{L} p_k w_k \big(\xi_k e^{w_k t} - (1-\xi_k) e^{-w_k t}\big)\Big)^2 = (M'(t))^2.$$

Therefore,

$$\frac{d^2 \log M(t)}{dt^2} = \frac{M''(t)M(t) - (M'(t))^2}{(M(t))^2} > 0,$$

so $\log M(t)$ is strictly convex. *Q.E.D.*

Lemma 3. *The range of* $\frac{d \log M(t)}{dt} = \frac{M'(t)}{M(t)}$ *is* $(-w_L, w_L)$.

Proof. We have already established that $\frac{d^2 \log M(t)}{dt^2} > 0$, so $\frac{M'(t)}{M(t)}$ is strictly increasing in t. Moreover,

$$\lim_{t \to -\infty} \frac{M'(t)}{M(t)} = \lim_{t \to -\infty} \frac{\sum_{k=1}^{L} p_k w_k \big(\xi_k e^{w_k t} - (1-\xi_k) e^{-w_k t}\big)}{\sum_{k=1}^{L} p_k \big(\xi_k e^{w_k t} + (1-\xi_k) e^{-w_k t}\big)} = -w_L,$$

and

$$\lim_{t \to \infty} \frac{M'(t)}{M(t)} = \lim_{t \to \infty} \frac{\sum_{k=1}^{L} p_k w_k \big(\xi_k e^{w_k t} - (1-\xi_k) e^{-w_k t}\big)}{\sum_{k=1}^{L} p_k \big(\xi_k e^{w_k t} + (1-\xi_k) e^{-w_k t}\big)} = w_L.$$

Therefore, the range of $\frac{M'(t)}{M(t)}$ is $(-w_L, w_L)$. *Q.E.D.*

Rate function

Denote the rate function as

$$I(x) := \sup_{t \in \mathbb{R}} tx - \log M(t).$$

Now let

$$g(x,t) = tx - \log M(t) = tx - \log\left[\sum_{k=1}^{L} p_k\Big(\xi_k e^{w_k t} + (1-\xi_k)e^{-w_k t}\Big)\right],$$

then $I(x) = \sup_{t\in\mathbb{R}} g(x,t)$.

When $x < -w_L$, we have $g(x,t) \to \infty$ as $t \to -\infty$, so $I(x) = \infty$.

When $x > w_L$, we have $g(x,t) \to \infty$ as $t \to \infty$, so $I(x) = \infty$.

When $x = -w_L$,

$$\begin{aligned} e^{g(x,t)} &= \frac{e^{-w_L t}}{\sum_{k=1}^{L} p_k\big(\xi_k e^{w_k t} + (1-\xi_k)e^{-w_k t}\big)} \\ &= \frac{1}{p_L\xi_L e^{2w_L t} + p_L(1-\xi_L) + \sum_{k=1}^{L-1} p_k\big(\xi_k e^{(w_k+w_L)t} + (1-\xi_k)e^{(w_L-w_k)t}\big)}. \end{aligned}$$

Thus, $e^{g(x,t)}$ is decreasing in t, so the supremum is attained as $t \to -\infty$, and

$$I(-w_L) = -\log\big(p_L(1-\xi_L)\big).$$

When $x = w_L$,

$$\begin{aligned} e^{g(x,t)} &= \frac{e^{w_L t}}{\sum_{k=1}^{L} p_k\big(\xi_k e^{w_k t} + (1-\xi_k)e^{-w_k t}\big)} \\ &= \frac{1}{p_L\xi_L + p_L(1-\xi_L)e^{-2w_L t} + \sum_{k=1}^{L-1} p_k\big(\xi_k e^{-(w_L-w_k)t} + (1-\xi_k)e^{-(w_L+w_k)t}\big)}. \end{aligned}$$

Thus, $e^{g(x,t)}$ is increasing in t, so the supremum is attained as $t \to \infty$, and

$$I(w_L) = -\log\big(p_L\xi_L\big).$$

When $x \in (-w_L, w_L)$, since $g(x,t)$ is concave in t, the supremum is attained at t^* satisfying

$$\frac{\partial g(x,t)}{\partial t} = x - \frac{M'(t)}{M(t)} = 0.$$

Since $\frac{M'(t)}{M(t)}$ is strictly increasing in t, there is a unique solution t^*. Thus, the rate function is

$$I(x) = \begin{cases} xt^* - \log M(t^*) & x \in (-w_L, w_L), \\ -\log\left(p_L(1-\xi_L)\right) & x = -w_L, \\ -\log\left(p_L\xi_L\right) & x = w_L, \\ \infty & x \notin [-w_L, w_L]. \end{cases}$$

Lemma 4. *t^* is continuous and increasing in x.*

Proof. $t^*(x)$ is solution to $x = \frac{M'(t)}{M(t)}$. $\frac{M'(t)}{M(t)}$ is continuous and strictly increasing in t, and the range is $(-w_L, w_L)$. So for each $x \in (-w_L, w_L)$, there is a unique $t^* \in \mathbb{R}$ such that $x = \frac{M'(t^*)}{M(t^*)}$. Also, $\lim_{t\to-\infty} \frac{M'(t)}{M(t)} = -w_m$, $\lim_{t\to\infty} \frac{M'(t)}{M(t)} = w_m$. Thus, $t^* \to -\infty$ as $x \to -w_m$, and $t^* \to \infty$ as $x \to w_m$.

By implicit function theorem, we have

$$\frac{dt^*}{dx} = \frac{1}{\frac{\partial^2 g(x,t)}{\partial t^2}} = \frac{1}{-\frac{M''(t)M(t)-(M'(t))^2}{(M(t))^2}} > 0.$$

So t^* is increasing in x. *Q.E.D.*

Lemma 5. *$I(x)$ is well-defined and continuous in x on $[-w_L, w_L]$.*

Proof. When $x \in (-w_L, w_L)$, $I(x)$ is well-defined since t^* is unique. Also, $I(x)$ is well-defined when $x = -w_L$ and $x = w_L$. Thus $I(x)$ is well-defined.

When $x \in (-w_L, w_L)$, we have t^* satisfies $x = \frac{M'(t^*)}{M(t^*)}$. Since t^* is a unique map between x

and t^*, we can write $I(x)$ as:

$$I(\frac{M'(t)}{M(t)}) = t \cdot \frac{M'(t)}{M(t)} - \log M(t).$$

Since $M'(t)$ and $M(t)$ are continuous in t, $I(\frac{M'(t)}{M(t)})$ is continuous in t. Thus, $I(x)$ is continuous on $(-w_L, w_L)$.

Let $t \to -\infty$,

$$\begin{aligned}
\lim_{x \to -w_L^+} e^{I(x)} &= \lim_{t \to -\infty} e^{t \cdot \frac{M'(t)}{M(t)} - \log M(t)} \\
&= \lim_{t \to -\infty} \frac{e^{t \cdot \frac{M'(t)}{M(t)}}}{M(t)} \\
&= \lim_{t \to -\infty} \frac{e^{t \cdot \frac{M'(t)}{M(t)}}}{\sum_{k=1}^{L} p_k(\xi_k e^{w_k t} + (1-\xi_k)e^{-w_k t})} \\
&= \lim_{t \to -\infty} \frac{1}{\sum_{k=1}^{L} p_k(\xi_k e^{(w_k - \frac{M'(t)}{M(t)})t} + (1-\xi_k)e^{(-w_k - \frac{M'(t)}{M(t)})t})}
\end{aligned}$$

Since $\lim_{t \to -\infty} \frac{M'(t)}{M(t)} = -w_L$, so for an $w^* \in (-w_L, -w_{L-1})$, there is a $\underline{t}$ such that when $t < \underline{t}$, $\frac{M'(t)}{M(t)} \in (-w_L, w^*)$.

Thus, we have

$$0 \leq \lim_{t \to -\infty} e^{(-w_k - \frac{M'(t)}{M(t)})t} \leq \lim_{t \to -\infty} e^{(-w_k - w^*)t} = 0, \quad k = 1, 2, ..., m-1.$$

Similarly,

$$0 \leq \lim_{t \to -\infty} e^{(w_k - \frac{M'(t)}{M(t)})t} \leq \lim_{t \to -\infty} e^{(w_k - w^*)t} = 0, \quad k = 1, 2, ..., m.$$

Thus, we have:

$$\lim_{x \to -w_L^+} e^{I(x)} = \lim_{t \to -\infty} \frac{1}{p_L(1-\xi_L)e^{(-w_L - \frac{M'(t)}{M(t)})t}}$$

The key is the term $(-w_L - \frac{M'(t)}{M(t)})t$.

$$\begin{aligned}
\lim_{t\to-\infty}(-w_L - \frac{M'(t)}{M(t)})t = & \lim_{t\to-\infty}\left(\frac{\sum_{k=1}^{L} p_k w_k((1-\xi_k)e^{-w_k t} - \xi_k e^{w_k t})}{\sum_{k=1}^{L} p_k(\xi_k e^{w_k t} + (1-\xi_k)e^{-w_k t})} - w_L\right)t \\
= & \lim_{t\to-\infty}\left(\frac{\sum_{k=1}^{L} p_k w_k((1-\xi_k)e^{(w_L-w_k)t} - \xi_k e^{(w_L+w_k)t})}{\sum_{k=1}^{L} p_k(\xi_k e^{(w_L+w_k)t} + (1-\xi_k)e^{(w_L-w_k)t})} - w_L\right)t \\
= & \lim_{t\to-\infty}\left(\frac{\sum_{k=1}^{L-1} p_k(w_k - w_L)(1-\xi_k)e^{(w_L-w_k)t} + \sum_{k=1}^{L} p_k(w_k + w_L)\xi_k e^{(w_L+w_k)t}}{\sum_{k=1}^{L} p_k(\xi_k e^{(w_L+w_k)t} + (1-\xi_k)e^{(w_L-w_k)t})}\right)t \\
= & \lim_{t\to-\infty}\frac{\sum_{k=1}^{L-1} p_k(w_k - w_L)(1-\xi_k)te^{(w_L-w_k)t} + \sum_{k=1}^{L} p_k(w_k + w_L)\xi_k te^{(w_L+w_k)t}}{\sum_{k=1}^{L} p_k(\xi_k e^{(w_L+w_k)t} + (1-\xi_k)e^{(w_L-w_k)t})}
\end{aligned}$$

All terms except $(1-\xi_L)e^{(w_L-w_L)t}$ will go to zero as $t \to -\infty$. Thus,

$$\lim_{t\to-\infty}\left(-w_L - \frac{M'(t)}{M(t)}\right)t = 0.$$

Thus, we have:

$$\lim_{x\to-w_L^+} e^{I(x)} = \frac{1}{p_L(1-\xi_L)}.$$

$I(x)$ is continuous at $-w_L$. Similarly, we can prove that $I(x)$ is continuous at w_L. Thus, $I(x)$ is continuous on $[-w_L, w_L]$. *Q.E.D.*

Lemma 6. *Let $x^0 = \sum_{k=1}^{L} p_k w_k(2\xi_k - 1))$. When $x < x^0$, I is decreasing; when $x > x^0$, I is increasing. I achieve its minimum at x^0.*

Proof. As before, we can write $I(x)$ as:

$$I(x) = g(x, t^*(x)).$$

By envelope theorem, we can get:

$$I'(x) = g_x(x, t^*(x)) = t^*(x)$$

Since $t^*(x)$ is strictly increasing in x, we need to find the point where $t^* = 0$. When $t^* = 0$, we have:

$$x^0 = \frac{M'(0)}{M(0)} = \frac{\sum_{k=1}^{L} p_k w_k(\xi_k - (1-\xi_k))}{\sum_{k=1}^{L} p_k w_k(\xi_k + 1 - \xi_k)} = \sum_{k=1}^{L} p_k w_k(2\xi_k - 1)$$

Thus, when $x < x^0$, $I'(x) = t^*(x) < 0$, so I is decreasing; when $x > x^0$, $I'(x) = t^*(x) > 0$, so I is increasing. Since I is continuous, I achieve its minimum at x^0. *Q.E.D.*

Applying Cramér's theorem

Cramer's theorem says that $\frac{S_n}{n}$ satisfies a large deviation principle with rate function $I(x)$.

Now we know $I(x)$ is decreasing when $x < x^0$, and $x^0 = \sum_{j=1}^{m} p_j w_j(2\xi_j - 1)) > 0$. Thus,

$$\inf_{x<0} I(x) = \inf_{x\le 0} I(x) = I(0).$$

If we have the optimal aggregation rule where the weights are $w_j = \log \frac{\xi_j}{1-\xi_j}$, when $x = 0$, $t^*(0) = -\frac{1}{2}$. This can be verified by:

$$\begin{aligned} M'(-\frac{1}{2}) = & \sum_{j=1}^{m} p_j w_j(\xi_j e^{w_j(-\frac{1}{2})} - (1-\xi_j)e^{-w_j(-\frac{1}{2})}) \\ = & \sum_{j=1}^{m} p_j w_j[\xi_j \left(\frac{\xi_j}{1-\xi_j}\right)^{-\frac{1}{2}} - (1-\xi_j)\left(\frac{1-\xi_j}{\xi_j}\right)^{-\frac{1}{2}}] \\ = & \sum_{j=1}^{m} p_j w_j(\sqrt{\xi_j(1-\xi_j)} - \sqrt{\xi_j(1-\xi_j)}) = 0 \end{aligned}$$

Thus

$$
\begin{aligned}
I(0) = & -\tfrac{1}{2}\cdot 0 - \log M(-\tfrac{1}{2}) \\
= & -\log \sum_{j=1}^{m} p_j(\xi e^{w_j(-\frac{1}{2})} + (1-\xi_j)e^{-w_j(-\frac{1}{2})}) \\
= & -\log \sum_{j=1}^{m} p_j[\xi_j \left(\frac{\xi_j}{1-\xi_j}\right)^{-\frac{1}{2}} + (1-\xi_j)\left(\frac{1-\xi_j}{\xi_j}\right)^{-\frac{1}{2}}] \\
= & -\log \left[\sum_{j=1}^{m} 2p_j\sqrt{\xi_j(1-\xi_j)}\right]
\end{aligned}
$$

By Cramer's theorem, we have

$$
\lim_{n\to\infty}\frac{1}{n}\log \Pr(S_n < 0) = \lim_{n\to\infty}\frac{1}{n}\log \Pr(S_n \leq 0) = \log\left[\sum_{j=1}^{m} 2p_j\sqrt{\xi_j(1-\xi_j)}\right]
$$

Now we know that $I(x)$ is decreasing when $x < x^0$, and

$$
x^0 = \sum_{k=1}^{L} p_k w_k(2\xi_k - 1) > 0.
$$

Thus,

$$
\inf_{x<0} I(x) = \inf_{x\leq 0} I(x) = I(0).
$$

Under the optimal aggregation rule with weights $w_k = \log\left(\frac{\xi_k}{1-\xi_k}\right)$, we have $t^*(0) = -\frac{1}{2}$. This can be verified by

$$
\begin{aligned}
M'\left(-\tfrac{1}{2}\right) = & \sum_{k=1}^{L} p_k w_k\left(\xi_k e^{w_k(-\frac{1}{2})} - (1-\xi_k)e^{-w_k(-\frac{1}{2})}\right) \\
= & \sum_{k=1}^{L} p_k w_k\left[\xi_k\left(\frac{\xi_k}{1-\xi_k}\right)^{-\frac{1}{2}} - (1-\xi_k)\left(\frac{1-\xi_k}{\xi_k}\right)^{-\frac{1}{2}}\right] \\
= & \sum_{k=1}^{L} p_k w_k\left(\sqrt{\xi_k(1-\xi_k)} - \sqrt{\xi_k(1-\xi_k)}\right) = 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I(0) = & -\tfrac{1}{2}\cdot 0 - \log M\left(-\tfrac{1}{2}\right) \\
= & -\log \sum_{k=1}^{L} p_k \Big(\xi_k e^{w_k(-\frac{1}{2})} + (1-\xi_k) e^{-w_k(-\frac{1}{2})}\Big) \\
= & -\log \sum_{k=1}^{L} p_k \left[\xi_k \left(\frac{\xi_k}{1-\xi_k}\right)^{-\frac{1}{2}} + (1-\xi_k)\left(\frac{1-\xi_k}{\xi_k}\right)^{-\frac{1}{2}}\right] \\
= & -\log \left[\sum_{k=1}^{L} 2p_k\sqrt{\xi_k(1-\xi_k)}\right].
\end{aligned}
$$

By Cramér's theorem,

$$
\lim_{n\to\infty}\frac{1}{n}\log \Pr(S_n<0) = \lim_{n\to\infty}\frac{1}{n}\log \Pr(S_n\le 0) = \log\left[\sum_{k=1}^{L} 2p_k\sqrt{\xi_k(1-\xi_k)}\right].
$$

Thus, we conclude that

$$
\phi = -\log\left[\sum_{k=1}^{L} 2p_k\sqrt{\xi_k(1-\xi_k)}\right].
$$

Q.E.D.